数理解析研究所講究録
第 1964 巻 2015 年 61-68

GOOD IDEALS AND ULRICH IDEALS IN NUMERICAL SMIGROUP RINGS

KEIICHI WATANABE
(COLLEGE OF HUMANITY AND SCIENCES, NIHON UNIVERSITY)

This is mainly a survey on good ideals and Ulrich ideals on semigroup rings of numerical semigroups. In the final part of this article, we give the characterization of Ulrich ideals on semigroup rings of almost symmetric numerical semigroups.

In my talk, $(A, \mathfrak{m})$ is a d-dimensional local domain (or $A = \oplus_{n\geq 0}A_n$ be a graded domain with $A_0 = k$, a field), which is Cohen-Macaulay.

Mainly I talk on the semigroup ring $k[[H]] = k[[t^a \mid a \in H]] \subset k[[t]]$ or $k[H] = k[t^a \mid a \in H]$, for a numerical semigroup H, which is Cohen-Macaulay of dimension 1. Let I be a (homogenous) $\mathfrak{m}$ primary ideal of A.

1. Fundamental Definitions and Examples.

Definition 1.1. Let $(A, \mathfrak{m}), I$ be as above.

(1) (minimal reduction) An ideal $Q \subset I$ is a *minimal reduction* of I, if Q is generated by d elements and $I^{r+1} = QI^r$ for some r. (if $A = k[H]$ and $I = (t^a, t^b, \ldots, t^c)$ with $a < b < \ldots < c$, then $Q = (t^a)$) We will always assume that $I \neq Q$.

(2) (stable ideal) I is called *stable* if $I^2 = IQ$ for a minimal reduction Q of I. Assume $d = 1$. If $I^2 = QI$ and if we put $Q = (a)$, then

$$B_I = a^{-1}I = \{\frac{x}{a} \mid x \in I\}$$

is a *ring* containing A and $B_I/A \cong I/Q$. In this manner, a stable ideal corresponds to an overring of A contained in the integral closure of A.

Definition 1.2. Let $(A, \mathfrak{m}), I$ be as above.

This paper is an announcement of our results and the detailed version will be submitted to somewhere else.

(1) (good ideal, [GIW]) I is *good* if $I^2 = QI$ and $Q : I = I$. In case $\dim A = 1$, I is good iff I is stable and $A : B_I = I$, where B_I is defined in 1.1. In this manner, there is one to one correspondence between *some* over rings of A and good ideals of A.

(2) (Ulrich ideal, [GOTWY]) I is an *Ulrich ideal* if (i) $I^2 = QI$ for some (any) minimal reduction Q and (ii) I/I^2 is a free A/I-module.

We give also the notion of Ulrich modules although we do not use this notion, because historically, Ulrich modules are defined before Ulrich ideals.

(3) Let I be an Ulrich ideal of A with minimal reduction Q. A maximal Cohen-Macaulay A module M is called an Ulrich module with respect to I if $QM = IM$ and M/IM is a free A/I module.

When we say that I is a good (resp. Ulrich) ideal, we always assume that $I \neq Q$.

Proposition 1.3. *Let A, I be as above with* $\dim A = d$ *and Q be a minimal reduction of A. We denote by $\mu_A(I)$ the number of minimal generators of I and $\ell_A(M)$ the length of the composition series of M for an A module M of finite length. We assume that I is stable. Then we have the following results.*

(1) *$e_A(I) = \ell_A(A/Q) \leq (\mu(I) - d + 1)\ell_A(A/I)$ and the equality holds if and only if I is an Ulrich ideal.*

(2) *If I is an Ulrich ideal, then I/Q is a free A/I module of rank $\mu_A(I) - d$. Hence if I is Ulrich, then I is good.*

(3) *If I is Ulrich, then* $\mathrm{type}(A) \geq (\mu(I) - 1)\mathrm{type}(A/I)$, *where* $\mathrm{type}(A)$ *means the type of A as a Cohen-Macuaylay ring. A Cohen-Macaulay ring A is Gorenstein if and only if* $\mathrm{type}(A) = 1$. *Hence if A is a Grenstein ring and I is a good ideal, then I is an Ulrich ideal if and only if $\mu_A(I) = d + 1$.*

Definition 1.4. Let $H \subset \mathbb{N} = \{0, 1, 2, \ldots\}$ be a numerical semigroup. Namely, H is closed under addition, $0 \in H$ and $\mathbb{N} \setminus H$ is finite. We denote $A = k[H]$ (or $k[[H]]$, where k is any field.

(1) We denote $e(H) = \min\{h \in H \mid h > 0\}$ and call *multiplicity* of H (or A). $\mathrm{emb}(H)$ is the number of minimal generators of H ($= \mathrm{emb}(A)$).

(2) $F(H) = \max\{x \in \mathbb{Z} \mid x \notin H\}$ is called the *Frobenius number* of H.

(3) $c(H) = F(H) + 1$ is called the *conductor* of H.

(4) $PF(H) = \{x \in \mathbb{Z} \mid \forall h \in H, h > 0, x + h \in H\}$ is called the set of *pseudo-Frobenius numbers.*

(5) $\mathrm{type}(H) = \sharp PF(H)$ is called the *type* of H. This coincides with the Cphen-Macaulay type of the semigroup ring $k[H]$ (or $k[[H]]$).

(6) H is called *symmetric* if $\mathrm{type}(H) = 1$. This condition is equivalent to say "for any $x \in \mathbb{Z}$, either $x \in H$ or $F(H) - x \in H$". H is symmetric if and only if $k[H]$ (or $k[[H]]$) is Gorenstein.

(7) H is a *complete intersecton* if the defining ideal of A is generated by $\mathrm{emb}(H) - 1$ elements. If H is a complete intersection, then H is symmetric and the converse is true if $\mathrm{emb}(H) \leq 3$.

(8) ([BF]) H is called *almost symmetric* if for every $x \in PF(H), x \neq F(H)$, $F(H) - x \in PF(H)$. This definition is different from the original one and was found by H. Nari ([Na]).

(9) H is called *pseudo-symmetric* if $F(H)$ is even and $\forall x \in \mathbb{Z}, x \neq F(H)/2$, either $x \in H$ or $F(H) - x \in H$. H is pseudo-symmetric if and only if H is almost symmetric and $\mathrm{type}(H) = 2$.

Theorem 1.5 (GIW). *Assume that A is a Gorenstein ring with* $\mathrm{dim} A = 1$. *Then*

(1) *There is one to one correspondence between the set of good ideals of A and Gorenstein over rings B of A such that $A \subset B \subset K$ (total quotient ring of A) and B is finite over A.*

(2) *There is one to one correspondence between the set of Ulrich ideals of A and Gorenstein over rings B of A such that $A \subset B \subset K$ (total quotient ring of A) and B/A is a principal A module.*

Question 1.6. If $\mathrm{dim} A = 1$ and A is not Gorenstein, what is the characterization of B_I with I a good ideal?

Example 1.7. Let k be any field, H be a numerical semigroup and $A = k[H]$ be the semigroup ring of H. We will denote $(a_1, a_2, \ldots, a_s)$ instead of $(t^{a_1}, t^{a_2}, \ldots, t^{a_s})$, where $(a_1, a_2, \ldots, a_s) = \cup_{i=1}^{s}(a_i + H)$. We denote $(a, a+1, \ldots, 2a-1)$ by $(a, \rightarrow)$.

(1) Let $\mathfrak{c} = (c(H), \rightarrow)$ be the conductor ideal of $k[H]$. Then $\mathfrak{c}$ is a good ideal. Hence any $k[H]$ has at least one good ideal.

(2) Let $H = \langle 4, 5, 6 \rangle$.

There is a sequence of semigroups
$\langle 4,5,6\rangle \subset \langle 4,5,6,7\rangle \subset \langle 2,5\rangle \subset \langle 2,3\rangle \subset \mathbb{N}$.
The corresponding decreasing sequence of ideals ($I = A : B$) is
$A \supset \mathfrak{m} \supset I_1 = (4,6) \supset I_2 = (6,8,9) \supset \mathfrak{c} = (8,\rightarrow)$,
where $\mathfrak{m}$ is not stable, I_1 is Ulrich and $I_3, \mathfrak{c}$ are good but not Ulrich.

(3) We will see that if $H = \langle a,b\rangle$ with a,b odd, then $A = k[H]$ has no homogeneous Ulrich ideal.

(4) (Taniguchi) Let $H = \langle 3,7\rangle$ and $A = k[[H]] = k[[t^3,t^7]]$. Then $I_c = (t^7 - ct^6, t^9)$ is Ulrich for $c \in k, c \neq 0$. Conversely, if I is an Ulrich ideal of A, then $I = I_c$ for some $c \neq 0$. On the other hand, $A = k[[t^3,t^5]]$ has no Ulrich ideals.

(5) K. Yoshida constructed Ulrich ideals of $k[[t^a,t^b]]$, where a,b odd and $b \geq 2a+1$.

Remark 1.8. ("Classical" Ulrich modules) In 1984 B. Ulrich investigated the Maximal Cohen-Macaulay (MCM) modules over CM local domain A with equality $\mu(M) = \mathrm{rank}(M)e(A)$, where $\mu(M)$ denotes the number of minimal generators of M and $e(A)$ is the multiplicity of A. We have $\leq$ always and thus Ulrich module is a MCM with most numbers of generators. Afterward, such modules are called "Ulrich modules". Even algebraic geometers (e.g. R. Hartshorne) studies "Ulrich Bundles" (vector bundles over projective varieties, the graded module associated to the bundle is an Umrich module.)

In [GOTWY] we investigated a relative version of this and we showed, for example, higher syzygy modules of an Ulrich ideal are Ulrich modules with respect to I. The classic Ulrich modules are Ulrich modules over $\mathfrak{m}$ in our language.

The theory of Ulrich ideals and good ideals have very rich results in dimension 2

Theorem 1.9. *([GOTWY]) Let $(A,\mathfrak{m})$ be a rational singularity of dimension 2. (In this case, every integrally closed ideal is stable [Li].)*

(1) *I is good if I is represented by minimal resolution of A. Hence the set of good ideals forms a semigroup and there are countable number of good ideals. (For example, if $A = k[X^r, X^{r-1}Y, \ldots, Y^r] \subset k[X,Y]$, the only good ideals on A are powers of $\mathfrak{m}$).*

(2) *The Ulrich ideals of A are completely classified using the geometric data of minimal resolution of A and each A has finite number of Ulrich ideals.*

If $\dim A = 2$ and A is not a rational singularity, the situation is quite different.

Theorem. [Okuma-W-Yoshida]

(1) If $A \cong k[[X,Y,Z]]/(X^3+Y^3+Z^3)$ and I is an Ulrich ideal of A, then $\ell_A(A/I) = 2$ and the set of Ulrich ideals of A corresponds to the points of elliptic curve $\{X^3+Y^3+Z^3=0\} \subset \mathbb{P}^2_k$. (The same holds for any simple elliptic singularity of multiplicity 3.)

(2) If A is a 2-dimensional normal Gorenstein ring with $p_g(A) = 1$ and multiplicity ≥ 5, then A has no Ulrich ideal.

(3) If A is normal and Gorenstein, there are infinitely many good ideals on A.

2. Classification of homogeneous Ulrich ideals in a numerical semigroup ring

In this section, we will try to determine good (reps. Ulrich) ideals of $k[H]$ generated by monomials (equivalently, homogeneous by natural grading of $k[H]$).

Henceforth, all ideals are homogeneous.

The first question will be the following.

Problem 2.1. *Given a numerical semigroup H, determine whether H has homogeneous Ulrich ideal or not.*

The first step to this question will be the following.

Theorem 2.2. *([GOTWY]) Let $A = k[t^a, t^b]$, with $(a,b) = 1$.*

(1) *If a, b are odd, then A has no Ulrich ideals.*

(2) *If $a = 2d$ and $b = 2l+1$, then the homogeneous Ulrich ideals of A are of the form $\{(t^{ia}, t^{db}) \mid 1 \leq i \leq l\}$.*

To proceed further, we will review on gluing of numerical semigroups. Note that if H is a complete intersection, then H is constructed by repeating the gluing from $H =< 1 >$. (cf. [RG], [De],[Wa]).

Definition 2.3. Let H_1, H_2 be numerical semigroups (including the case $H_i =< 1 >= \mathbb{N}$), $m_i \in H_i$ $(i = 1, 2)$, which are not one of the minimal generators and assume $(m_1, m_2) = 1$. Then $H = \langle n_2 H_1, n_1 H_2 \rangle$ is called a *gluing* of H_1 and H_2.

Proposition 2.4. *Let H be a gluing of H_1 and H_2 as above. Then*

(1) $\mathrm{type}(H) = \mathrm{type}(H_1) \cdot \mathrm{type}(H_2)$. *Hence H is symmetric if and only if both H_i $(i = 1, 2)$ are symmetric.*

(2) *([Na]) H is not almost symmetric unless H is symmetric.*

(3) *([Nu]) There is a natural flat ring homomorphism $k[H_i] \to k[H]$. If I is an Ulrich ideal of $k[H_1]$, say, then $Ik[H]$ is an Ulrich ideal of $k[H]$. In this case, we say "$Ik[H]$ is lifted from $k[H_1]$."*

To determine Ulrich ideals in the case H is symmetric, the following Theorem is fundamental.

Theorem 2.5. *([GOTWY]) Let H be a symmetric semigroup and $A = k[H] \subset k[t]$ be its semigroup ring. Then $I = (t^a, t^b)$ $(a < b)$ is an Ulrich ideal of A if and only if $b - a \notin H$, $2(b - a) \in H$ and $\langle H, b - a\rangle$ is symmetric. Conversely, if we take $x \in \mathbb{N}$ such that $x \notin H, 2x \in H$ and $\langle H, x\rangle$ is symmetric, then $I = (t^a, t^b)$ is an Ulrich ideal of A, where $a = min\{h \in H \mid x + h \in H\}$.*

Let $A = k[H]$. We denote by χ_A^g the set of Ulrich ideals generated by homogeneous elements. When H is symmetric with $\mathrm{emb}(H) = 3$, then χ_A^g is completely determined by T. Numata. We use the fact that 3 generated symmetric semigroups are complete intersections and obtained by gluing.

Theorem 2.6. *([Nu]) Let $H = \langle a, b, c\rangle$ be a symmetric numerical semigroup and assume that $H = \langle d\langle a', b'\rangle, c\rangle$. We set $R = k[[H]]$, $H_1 = \langle a', b'\rangle$ and $R_1 = k[[H_1]]$. Then the following assertions hold true.*

(1) *If d and c are odd, then $\#\chi_R^g = \#\chi_{R_1}^g$ and every Ulrich ideal of R is lifted from R_1. In particular,*

(2) *If a, b, c are odd, then $\chi_R^g = \emptyset$.*

In the following, we assume that a' and b' are odd (equivalently, $\chi_{R_1}^g = \emptyset$).

(3) *If d is odd and c is even, then*
   (i) *$\chi_R^g \neq \emptyset$ if and only if $H + \langle c/2\rangle$ is symmetric.*
   (ii) *if $\chi_R^g \neq \emptyset$, then $\#\chi_R^g = (d - 1)/2$.*

(4) *If d is even and c is odd, then $H + \langle da'/2\rangle$ or $H + \langle db'/2\rangle$ is symmetric. In particular, $\chi_R^g \neq \emptyset$.*

3. Ulrich ideals of $k[H]$, where H is not symmetric

We begin by an easy example.

Example 3.1. By the definition of Ulrich ideals, the maximal ideal $\mathfrak{m}$ is a Ulrich ideal if and only if $\mathfrak{m}$ is stable, or equivalently, A has maximal embedding dimension $(\text{emb}(A) = e(A) + 1 - \dim A)$.

After the conference in February, we got a characterization of homogenous Ulrich ideals in the case of almost symmetric semigoups.

Theorem 3.2. *Let H be an almost symmetric numerical semigroup and $A = k[H]$. If I is an Ulrich ideal of A, then $I = \mathfrak{m}$. Hence if A has not maximal embedding dimension, then A has no Ulrich ideal.*

Remark 3.3. After the author presented this theorem at Goto seminar in Meiji University, S. Goto extended this result to almost Gorenstein rings of dimension 1. Hence we can drop the assumption "homogeneous" in the Theorem above. Also, the assumption "almost homogeneous" can be weakened to the following;

For some $x, x' \in PF(H)$ (possibly $x = x'$), $x + x' \notin H$.

Question 3.4. The author knows no examples of Ulrich ideals for $k[H]$ when H is generated by 3 elements and not symmetric (with $e(H) \geq 4$) or H is generated by 4 elements, symmetric and not a complete intersection.

Acknowledgement. The author thanks T. Numata and S. Goto for valuable discissions. Also, he is grateful to T. Matsuoka for showing examples of Ulrich ideals of $k[H]$, where H is neither symmetric nor almost symmetric.

References

[BF] V. Barucci, R. Fröberg, *One-dimensional almost Gorenstein rings*, J. Algebra, **188** (1997), 418-442.

[De] D. Delorme, *Sous-monoïdes d'intersection complète de* $\mathbb{N}$, Ann. Sci. Ècole Norm. Sup. **9** (1976), 145-154.

[GIW] S. Goto, S. Iai, and K. Watanabe, *Good ideals in Gorenstein local rings*, Trans. Amer. Math. Soc., **353** 2000, 2309–2346.

[GOTWY] S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe and K. Yoshida, *Ulrich ideals and modules,* To appear in Mathematical Proceedings of the Cambridge Philosophical Society, 2013

[GIW] S. Goto, S. Iai and K. Watanabe, Good ideals in Gorenstein local rings, Trans. A.M.S. 353 (2001), 2309-2346

[Li] J. Lipman, *Rational singularities with applications to algebraic surfaces and unique factorization*, Inst. Hautes Études Sci. Publ. Math. **36**, (1969), 195–279.

[Na] H. Nari, *Symmetries on almost symmetric numerical semigroups*, Semigroup Forum **86** (2013), no. 1, 140 - 154.

[Nu] Takahiro Numata, Ulrich ideals of certain Gorenstein numerical semigroup rings, To appear in Journal of Commutative Algebra.

[OWY] T. Okuma, K.-i. Watanabe and K. Yoshida, Good ideals and p_g -ideals in two-dimensional normal singularities, submitted, arXiv, mathAC1407.1595.

[RG] J. C. Rosales and P. A. García-Sánchez, *Numerical semigroups*, Springer Developments in Mathematics, Volume **20** (2009).

[Wa] K - i. Watanabe, *Some examples of one dimensional Gorenstein domains*, Nagoya Math. J. **49** (1973), 101-109.